

ON ADDITIVITY OF LOCAL ENTROPY UNDER FLAT EXTENSIONS

MAHDI MAJIDI-ZOLBANIN

ABSTRACT. Let $f: (R, \mathfrak{m}) \rightarrow S$ be a local homomorphism of Noetherian local rings. Consider two endomorphisms of *finite length* (i.e., with zero-dimensional closed fibers) $\varphi: R \rightarrow R$ and $\psi: S \rightarrow S$, satisfying $\psi \circ f = f \circ \varphi$. Then ψ induces a finite length endomorphism $\bar{\psi}: S/f(\mathfrak{m})S \rightarrow S/f(\mathfrak{m})S$. When f is flat, under the assumption that S is Cohen-Macaulay we prove an additivity formula: $h_{\text{loc}}(\psi) = h_{\text{loc}}(\varphi) + h_{\text{loc}}(\bar{\psi})$ for *local entropy*.

1. INTRODUCTION

All rings in this note are assumed to be Noetherian, local, commutative and with identity element 1.

The notion of *local entropy* associated with an endomorphism of *finite length* of a Noetherian local ring was introduced in [1]. We recall a few definitions and results from [1].

Definition 1 ([1, Definition 1]). A local homomorphism $f: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ of Noetherian local rings is said to be of *finite length*, if one of the following equivalent conditions holds:

- a) $f(\mathfrak{m})S$ is \mathfrak{n} -primary;
- b) The closed fiber of f has dimension zero;
- c) If \mathfrak{p} is a prime ideal of S such that $f^{-1}(\mathfrak{p}) = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{n}$;
- d) If \mathfrak{q} is any \mathfrak{m} -primary ideal of R , then $f(\mathfrak{q})S$ is \mathfrak{n} -primary.

Definition 2 ([1, Definition 5]). A *local algebraic dynamical system* (R, φ) consists of a Noetherian local ring R and an endomorphism of finite length $\varphi: R \rightarrow R$. By a *morphism* $f: (R, \varphi) \rightarrow (S, \psi)$ between two local algebraic dynamical systems we mean a local homomorphism $f: R \rightarrow S$ satisfying $\psi \circ f = f \circ \varphi$.

Theorem ([1, Theorem 1]). Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Write φ^n for the n -fold composition of φ with itself and let $\text{length}_R(-)$ denote the length of an R -module of finite length. Then the limit

$$h_{\text{loc}}(\varphi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log(\text{length}_R(R/\varphi^n(\mathfrak{m})R))$$

exists and is a nonnegative real number.

The invariant $h_{\text{loc}}(\varphi)$ is called the *local entropy* of φ . Local entropy can be calculated using any \mathfrak{m} -primary ideal:

Date: September 7, 2014.

2010 *Mathematics Subject Classification.* Primary 13B40, 14B25, 13B10; Secondary 37P99.

Key words and phrases. Local entropy, Flat extensions, Algebraic dynamics.

The author received funding from C³IRG (round 10) grant provided by the City University of New York.

Lemma 1. Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. If \mathfrak{q} is an \mathfrak{m} -primary ideal of R , then

$$h_{\text{loc}}(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log (\text{length}_R(R/\varphi^n(\mathfrak{q})R)).$$

Proof. This is a particular case of [1, Proposition 18]. We refer the reader to *loc. cit.* for a proof. \square

This paper is concerned with the following question asked by Craig Huneke:

Question 1. Let $f: (R, \mathfrak{m}, \varphi) \rightarrow (S, \mathfrak{n}, \psi)$ be a morphism between two local algebraic dynamical systems. Then by definition of morphism, $\psi \circ f = f \circ \varphi$. The ideal $f(\mathfrak{m})S$ is quickly seen to be ψ -stable (i.e., $\psi(f(\mathfrak{m})S) \subseteq f(\mathfrak{m})S$). Thus, ψ induces a finite length endomorphism $\bar{\psi}: S/f(\mathfrak{m})S \rightarrow S/f(\mathfrak{m})S$. If f is flat, does it hold that

$$(1.1) \quad h_{\text{loc}}(\psi) = h_{\text{loc}}(\varphi) + h_{\text{loc}}(\bar{\psi})?$$

We mention two cases in which the answer to Question 1 is affirmative: (1) When $\dim R = \dim S$, the question has an affirmative answer, as shown in [1, Proposition 20]. (2) By [1, Theorem 1] the local entropy of the Frobenius endomorphism of a local ring of positive prime characteristic p and of dimension d is equal to $d \cdot \log p$. Hence, when R and S are of positive prime characteristic p , and φ and ψ are their Frobenius endomorphisms, respectively, then Equation 1.1 reduces to

$$(\dim S) \cdot \log p = (\dim R) \cdot \log p + (\dim S/f(\mathfrak{m})S) \cdot \log p,$$

which holds, since f is flat (see, e.g., [2, Theorem 15.1]). The aim of this paper is to give an affirmative answer to Question 1 in the particular case when S is Cohen-Macaulay. The question remains open in the general (non Cohen-Macaulay) case.

2. MAIN RESULT

We will use the following Flatness Criterion in the proof of our main result, Theorem 1, as well as in Example 1. The reader can find a proof of this result in [2, Corollary to Theorem 22.5].

Theorem (Flatness Criterion). *Let $f: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings and let M be a finite S -module. For $y_1, \dots, y_n \in \mathfrak{n}$ write \bar{y}_i for the images of y_i in $S/f(\mathfrak{m})S$. Then the following conditions are equivalent:*

- a) y_1, \dots, y_n is an M -regular sequence and $M/\sum_1^n y_i M$ is flat over R ;
- b) $\bar{y}_1, \dots, \bar{y}_n$ is an $(M/f(\mathfrak{m})M)$ -regular sequence and M is flat over R .

We will also need the following elementary statement:

Proposition 1. Let $f: (R, \mathfrak{m}) \rightarrow S$ be a local homomorphism of finite length of Noetherian local rings. Let M be an R -module of finite length. Then

- a) $M \otimes_R S$ is of finite length as an S -module.
- b) $\text{length}_S(M \otimes_R S) \leq \text{length}_R(M) \cdot \text{length}_S(S/f(\mathfrak{m})S)$.
- c) If f is flat, then $\text{length}_S(M \otimes_R S) = \text{length}_R(M) \cdot \text{length}_S(S/f(\mathfrak{m})S)$.

Proof. By induction on $\text{length}_R(M)$. \square

We begin with showing that without assuming flatness, an inequality will still hold between local entropies:

Proposition 2. Let $f: (R, \mathfrak{m}, \varphi) \rightarrow (S, \mathfrak{n}, \psi)$ be a morphism between two local algebraic dynamical systems and let $\bar{\psi}: S/f(\mathfrak{m})S \rightarrow S/f(\mathfrak{m})S$ be the endomorphism induced by ψ . Then the following inequality holds:

$$h_{\text{loc}}(\psi) \leq h_{\text{loc}}(\varphi) + h_{\text{loc}}(\bar{\psi}).$$

Proof. The composition of maps $R \xrightarrow{f} S \rightarrow S/\psi^n(\mathfrak{n})S$ gives a local homomorphism of finite length $R \rightarrow S/\psi^n(\mathfrak{n})S$. Applying Proposition 1, we can write:

$$\begin{aligned} \text{length}_S(S/\psi^n(\mathfrak{n})S) &= \text{length}_S((R/\varphi^n(\mathfrak{m})R) \otimes_R (S/\psi^n(\mathfrak{n})S)) \\ &\leq \text{length}_R(R/\varphi^n(\mathfrak{m})R) \cdot \text{length}_S(S/(f(\mathfrak{m})S + \psi^n(\mathfrak{n})S)). \end{aligned}$$

We obtain the desired inequality by applying logarithm, dividing by n and taking limits as $n \rightarrow \infty$. \square

Remark 3. In a Cohen-Macaulay Noetherian local ring of dimension d , a sequence of d elements in the maximal ideal form a system of parameters if and only if they form a (maximal) regular sequence. We will use this fact frequently in our proof of Theorem 1. The reader can find a proof of this fact in [2, Theorem 17.4].

We now give an affirmative answer to Question 1 in the particular case when S is Cohen-Macaulay:

Theorem 1. Let $f: (R, \mathfrak{m}, \varphi) \rightarrow (S, \mathfrak{n}, \psi)$ be a flat morphism between two local algebraic dynamical systems and let $\bar{\psi}: S/f(\mathfrak{m})S \rightarrow S/f(\mathfrak{m})S$ be the endomorphism induced by ψ . If S is Cohen-Macaulay, then

$$(2.1) \quad h_{\text{loc}}(\psi) = h_{\text{loc}}(\varphi) + h_{\text{loc}}(\bar{\psi}).$$

Proof. Since f is flat, the Cohen-Macaulayness of S implies that the rings R and $S/f(\mathfrak{m})S$ are also Cohen-Macaulay (see, e.g., [2, Corollary to Theorem 23.3]). Since $S/f(\mathfrak{m})S$ is Cohen-Macaulay, there exists a (non-unique) sequence of elements $y_1, \dots, y_{d'} \in \mathfrak{n}$ of length $d' = \dim(S/f(\mathfrak{m})S)$, whose images in $S/f(\mathfrak{m})S$ form an $(S/f(\mathfrak{m})S)$ -regular sequence. Note that by the Flatness Criterion mentioned earlier, $y_1, \dots, y_{d'}$ is an S -regular sequence. Let $\mathfrak{q}' \subset S$ be the ideal generated by $y_1, \dots, y_{d'}$. We claim that for any integer $n \geq 0$, the ring $S/\psi^n(\mathfrak{q}')S$ is flat over R via the composition of maps

$$(2.2) \quad R \xrightarrow{f} S \rightarrow S/\psi^n(\mathfrak{q}')S.$$

Since $R \xrightarrow{f} S$ is flat, the claim will be established by the Flatness Criterion, if we can show that the images of $\psi^n(y_1), \dots, \psi^n(y_{d'})$ in $S/f(\mathfrak{m})S$ form an $(S/f(\mathfrak{m})S)$ -regular sequence. These images coincide with elements

$$\bar{\psi}^n(\bar{y}_1), \dots, \bar{\psi}^n(\bar{y}_{d'}),$$

where \bar{y}_i is the image of y_i in $S/f(\mathfrak{m})S$. That $\bar{\psi}^n(\bar{y}_1), \dots, \bar{\psi}^n(\bar{y}_{d'})$ is an $(S/f(\mathfrak{m})S)$ -regular sequence is an immediate consequence of Remark 3, the fact that $\bar{y}_1, \dots, \bar{y}_{d'}$ is a maximal $(S/f(\mathfrak{m})S)$ -regular sequence, and the fact that $\bar{\psi}^n$ is an endomorphism of finite length of $S/f(\mathfrak{m})S$ (hence, the image under $\bar{\psi}^n$ of any system of parameters is again a system of parameters in $S/f(\mathfrak{m})S$).

Now let $x_1, \dots, x_d \in \mathfrak{m}$ be an R -regular sequence of length $d = \dim R$ and let $\mathfrak{q} \subset R$ be the ideal generated by x_1, \dots, x_d . By Remark 3, \mathfrak{q} is generated by a system of parameters in R . By the flatness of $S/\mathfrak{q}'S$ over R via the composition of maps shown in 2.2 (taking $n = 0$), the images of $f(x_1), \dots, f(x_d)$ in $S/\mathfrak{q}'S$ form an

$(S/\mathfrak{q}'S)$ -regular sequence. This means $y_1, \dots, y_{d'}, f(x_1), \dots, f(x_d)$ is an S -regular sequence. Moreover, since f is flat,

$$d + d' = \dim R + \dim(S/f(\mathfrak{m})S) = \dim S$$

(see, e.g., [2, Theorem 15.1]). Hence, $\{y_1, \dots, y_{d'}, f(x_1), \dots, f(x_d)\}$ is a system of parameters in S , by Remark 3. Let $\mathfrak{Q} \subset S$ be the ideal generated by

$$y_1, \dots, y_{d'}, f(x_1), \dots, f(x_d).$$

We note that for any integer $n \geq 0$

$$(2.3) \quad \frac{R}{\varphi^n(\mathfrak{q})R} \otimes_R \frac{S}{\psi^n(\mathfrak{q}')S} \cong \frac{S}{f(\varphi^n(\mathfrak{q}))S + \psi^n(\mathfrak{q}')S} \cong \frac{S}{\psi^n(\mathfrak{Q})S},$$

where the last isomorphism quickly follows from the fact that $\psi \circ f = f \circ \varphi$. Since $S/\psi^n(\mathfrak{q}')S$ is flat over R and

$$\dim(S/\psi^n(\mathfrak{q}')S) = \dim S - d' = \dim S - \dim(S/f(\mathfrak{m})S) = \dim R,$$

the homomorphism $R \rightarrow S/\psi^n(\mathfrak{q}')S$ obtained by composing the maps given in 2.2 is in fact, of finite length. Hence, Proposition 1-c) applies and from 2.3 we obtain

$$\begin{aligned} \text{length}_S(S/\psi^n(\mathfrak{Q})S) &= \text{length}_S\left(\frac{R}{\varphi^n(\mathfrak{q})R} \otimes_R \frac{S}{\psi^n(\mathfrak{q}')S}\right) \\ &= \text{length}_R(R/\varphi^n(\mathfrak{q})R) \cdot \text{length}_S(S/[f(\mathfrak{m})S + \psi^n(\mathfrak{q}')S]). \end{aligned}$$

After applying logarithm to both sides, dividing by n and taking limits as $n \rightarrow \infty$, we obtain the desired Equation 2.1, by Lemma 1. \square

Example 1. In this example we will apply Theorem 1 to calculate local entropy of a specific endomorphism. Consider the endomorphism of the ring $(\mathbb{Z}/2\mathbb{Z})[[x, y, w, s]]$ that maps x, y, w and s to $x^3 + s^3, y^3, w^5 + x^2$ and xs^2 , respectively. This endomorphism is of finite length, because if \mathfrak{p} is a minimal prime ideal of $(x^3 + s^3, y^3, w^5 + x^2, xs^2)$, then as one can quickly see, $\mathfrak{p} = (x, y, w, s)$. One can also verify quickly that the ideal $(s^6, y^3 + x^2)$ is stable under this endomorphism. Thus, we obtain an induced ring endomorphism:

$$\psi: \frac{(\mathbb{Z}/2\mathbb{Z})[[x, y, w, s]]}{(s^6, y^3 + x^2)} \rightarrow \frac{(\mathbb{Z}/2\mathbb{Z})[[x, y, w, s]]}{(s^6, y^3 + x^2)}.$$

To abbreviate notation we will write S for the ring $(\mathbb{Z}/2\mathbb{Z})[[x, y, w, s]]/(s^6, y^3 + x^2)$. Our goal in this example is to calculate $h_{\text{loc}}(\psi)$, the local entropy of ψ . We will do this by constructing a flat homomorphism into the ring S and then using Theorem 1. Note that S is Cohen-Macaulay by virtue of being a complete intersection. Let $R = (\mathbb{Z}/2\mathbb{Z})[[y]]$ and let $\varphi: R \rightarrow R$ be the endomorphism that maps y to y^3 . To define a homomorphism $f: R \rightarrow S$ set $f(y) = y$ and then extend it linearly to all of R . It is evident that $f \circ \varphi = \psi \circ f$. From the Flatness Criterion that was stated earlier, it quickly follows that f is flat. Hence, by Theorem 1

$$\begin{aligned} h_{\text{loc}}(\psi) &= h_{\text{loc}}(\varphi) + h_{\text{loc}}(\overline{\psi}) \\ &= \log 3 + h_{\text{loc}}(\overline{\psi}), \end{aligned}$$

where as usual $\overline{\psi}$ is the endomorphism induced by ψ on S/yS . (That $h_{\text{loc}}(\varphi) = \log 3$ can be calculated quickly, using the definition of local entropy.) The ring S/yS is isomorphic to $S' := (\mathbb{Z}/2\mathbb{Z})[[x, w, s]]/(s^6, x^2)$ and $\overline{\psi}$ maps x, w and s to s^3, w^5 and xs^2 , respectively. To calculate $h_{\text{loc}}(\overline{\psi})$, we construct another flat homomorphism,

this time into S' . Let $R' := (\mathbb{Z}/2\mathbb{Z})[[w]]$ and let $\varphi': R' \rightarrow R'$ be the endomorphism that maps w to w^5 . To define a homomorphism $f': R' \rightarrow S'$ set $f'(w) = w$ and then extend it linearly to all of R' . Again it is evident that $f \circ \varphi = \psi \circ f$ and the flatness of f' quickly follows from the Flatness Criterion that was stated earlier. By Theorem 1, and using the fact that the local entropy of an endomorphism of a zero-dimensional local ring is zero ([1, Corollary 16]), we obtain $h_{\text{loc}}(\overline{\psi}) = \log 5$. Hence, $h_{\text{loc}}(\psi) = \log 3 + \log 5$.

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DEPARTMENT OF MATHEMATICS, LAGUARDIA COMMUNITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, 31-10 THOMSON AVENUE, LONG ISLAND CITY, NY 11101

E-mail address: mmajidi-zolbanin@lagcc.cuny.edu